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Abstract

This article is devoted to prove a stability result for two independent coefficients for a 2×2 nonlinear parabolic system with only one observation. The main idea to obtain this result is to use a modified form of the Carleman estimate
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1 Introduction and main result

This paper is an improvement of the work [4] in the sense that we determine two independent coefficients with the observation of only one component in a nonlinear 2×2 parabolic system. The system of coupled linear reaction-diffusion-convection equations had been recently widely studied in [1] and [2] with applications in inverse area but also in control domains.
Several works concern inverse problems associated with linear and nonlinear parabolic equations (see [13], [15], [3], [7], [8], [17], [5], ...) but few concern system of nonlinear parabolic equations. Such systems arise in biological,

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ecological domain or combustion and chemical reactions (see [16], [18]).
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of $\mathbb{R}^n$ with $n \leq 3$ and $\omega \subset \Omega$ a non empty subset. We denote by ν the outward unit normal to Ω on $\Gamma = \partial\Omega$ assumed to be of class $\mathcal{C}^1$. Let $T > 0$ and $t_0 \in (0, T)$. We shall use the following notations $Q_0 = \Omega \times (0, T)$, $Q = \Omega \times (t_0, T)$, $Q_\omega = \omega \times (t_0, T)$, $\Sigma = \Gamma \times (t_0, T)$ and $\Sigma_0 = \Gamma \times (0, T)$. We consider the following 2×2 reaction-diffusion system:

$$\begin{cases} \partial_t U = \Delta U + a_{11}(x)U + a_{12}(x)V + a_{13}(x)f(U,V) & \text{in} \quad Q_0, \\ \partial_t V = \Delta V + a_{21}(x)U + a_{22}(x)V & \text{in} \quad Q_0, \\ U(x,t) = k_1(x,t), \ V(x,t) = k_2(x,t) & \text{on} \quad \Sigma_0, \\ U(x,0) = U_0 \quad \text{and} \quad V(x,0) = V_0 & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where, the function f is assumed to be Lipschitz with respect the two variables U and V.
Uniqueness and existence results for initial boundary value problem for such systems can be found in [14].
Throughout this paper, we consider the following set

$$\Lambda(R) = \{\Phi \in L^\infty(\Omega); \ \|\Phi\|_{L^\infty(\Omega)} \leqslant R\},$$

where R is a given positive constant.
For $t_0 \in (0, T)$, we denote $T' = \frac{t_0+T}{2}$. Let (U, V) (resp. $(\widetilde{U}, \widetilde{V})$) be solution of (1) associated to $(a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, k_1, k_2, U_0, V_0)$ (resp. $(a_{11}, a_{12}, \widetilde{a}_{13}, \widetilde{a}_{21}, a_{22}, k_1, k_2, U_0, V_0)$) satisfying some regularity and positivity properties:

Assumption 1.1. *1. For $i = 1, 2$, $j = 1, 2, 3$, a_{ij}, $\widetilde{a}_{13}$ and $\widetilde{a}_{21} \in \Lambda(R)$.*

2. There exist constants $r_1 > 0$ and $a_0 > 0$ such that
$\widetilde{U}_0 \geq r_1$, $\widetilde{V}_0 \geq 0$, $a_{11}r_1 + a_{12}\widetilde{V}_0 + \widetilde{a}_{13}\, f(r_1, \widetilde{V}_0) \geq 0$,
$a_{21} \geq a_0$, $\widetilde{a}_{21} \geq a_0$, $k_1 \geq r_1$ *and* $k_2 \geq 0$.

Such assumptions allow us to state that the function $\widetilde{U}$ satisfies $|\widetilde{U}(x, T')| \geq r_1 > 0$ in Ω (see [19]).

Assumption 1.2. *1. The function f checks a generalized Lipschitz property in the following sense : $\exists\ C > 0$, such that*
$|\partial_t f(U,V) - \partial_t f(\widetilde{U}, \widetilde{V})| \leq C \left(|U - \widetilde{U}| + |V - \widetilde{V}| + |(U - \widetilde{U})_t| + |(V - \widetilde{V})_t| \right).$

2. $\exists\ r_2 > 0$ such that $f(\widetilde{U}, \widetilde{V})(T', x) \geq r_2 > 0$ in Ω.

3. $\partial_t f(U,V) \in L^2((0,T); H^2(\Omega))$.

This set of functions is not empty and contains, in particular, a large class of semilinear terms (e.g. $f(U,V) = U^\alpha V^\beta$ with α and β non negative constants in chemical reactions).
The main result is the following Theorem:

Theorem 1.3. *Let ω be a subdomain of an open set Ω of $\mathbb{R}^n$. We suppose that Assumptions 1.1 and 1.2 are checked and $(U,V)(\cdot,T') = (\widetilde{U},\widetilde{V})(\cdot,T')$. Furthermore, we assume that U_0, V_0 in $H^2(\Omega)$. Then there exists a constant $C = C(\Omega,\omega,a_0,t_0,T,r_1,r_2,R) > 0$ such that*

$$\|a_{21} - \widetilde{a}_{21}\|^2_{L^2(\Omega)} + \|a_{13} - \widetilde{a}_{13}\|^2_{L^2(\Omega)} \leq C\|\partial_t V - \partial_t \widetilde{V}\|^2_{L^2(Q_\omega)}.$$

In [4], for a coupled linear reaction diffusion system, we prove a stability result for one coefficient with only one observation. Note that our method does not work if the system (1) contains a non linear term in each equation. The novelty in this paper is the identification of two coefficients with only one observation for a nonlinear system. The main tool is a Carleman estimate established in [4] which is adapted, using a "shift" of the large parameters, to recover two independent coefficients, one in each equation of (1).
The paper is organized as follows: In section 2, we give the modified Carleman estimate for a reaction-diffusion system with only one observation. Then using this modified Carleman estimate, we prove in section 3 a stability result for two coefficients with the observation of only one component.

2 Carleman estimate

At first, we recall the general form of the Carleman estimate associated to the operator $\partial_t q - \Delta q$ (see [9], [11], [12]). Let $\omega' \Subset \omega \Subset \Omega$ and let $\widetilde{\beta}$ be a $\mathcal{C}^2(\Omega)$ function such that

$$\widetilde{\beta} > 0, \text{ in } \Omega,\ \widetilde{\beta} = 0 \text{ on } \partial\Omega,\ \min\{|\nabla\widetilde{\beta}(x)|,\ x \in \overline{\Omega \setminus \omega'}\} > 0 \quad \text{and} \quad \partial_\nu \widetilde{\beta} < 0 \text{ on } \partial\Omega.$$

Then, we define $\beta = \widetilde{\beta} + K$ with $K = m\|\widetilde{\beta}\|_\infty$ and $m > 1$. For $\lambda > 0$ and $t \in (t_0,T)$, we define the following weight functions (see [10])

$$\varphi(x,t) = \frac{e^{\lambda\beta(x)}}{(t-t_0)(T-t)}, \qquad \eta(x,t) = \frac{e^{2\lambda K} - e^{\lambda\beta(x)}}{(t-t_0)(T-t)}.$$

We then have the following Carleman estimate:

Theorem 2.1. *Let $\tau \in \mathbb{R}$. Then there exist $\lambda_0 = \lambda_0(\Omega, \omega) \geq 0$, $s_0 = s_0(\lambda_0, T, \tau) > 0$ and a positive constant $C_0 = C_0(\Omega, \omega, \tau)$ such that, for any $\lambda \geq \lambda_0$ and any $s \geq s_0$, the following estimate holds:*

$$I(\tau, q) \leq C_0 \left[\iint_{Q_\omega} e^{-2s\eta} \lambda^4 (s\varphi)^{\tau+3} |q|^2 \ dx \ dt + \iint_Q e^{-2s\eta} (s\varphi)^\tau |\partial_t q - \Delta q|^2 \ dx \ dt \right], \quad (2)$$

where

$$I(\tau,\, q) = \iint_Q e^{-2s\eta} (s\varphi)^{\tau-1} (|\partial_t q|^2 + |\Delta q|^2) \ dx \ dt + \lambda^2 \iint_Q e^{-2s\eta} (s\varphi)^{\tau+1} |\nabla q|^2 \ dx \ dt$$
$$+ \lambda^4 \iint_Q e^{-2s\eta} (s\varphi)^{\tau+3} |q|^2 \ dx \ dt$$

Remark 1. *If we denote*

$$M_1^{(\tau)} \psi = -\Delta\psi - s^2 \lambda^2 \varphi^2 |\nabla\beta|^2 \psi - (\frac{\tau}{2} - s\partial_t \eta)\psi,$$

and

$$M_2^{(\tau)} \psi = \partial_t \psi + 2s\lambda(\varphi + \frac{\tau}{2}) \nabla\beta . \nabla\psi,$$

with $\psi = e^{-s\eta} \varphi^{\frac{\tau}{2}} q$, the Carleman estimate (2) also gives an upper bound of $\|M_1^{(\tau)} \psi)\|^2_{L^2(Q)} + \|M_2^{(\tau)} \psi\|^2_{L^2(Q)}$ (see [9]).

We assume that $a_{11}, a_{12}, a_{21}, a_{22} \in \Lambda(R)$, $a_{21} \geq a_0 > 0$ and we consider the following system:

$$\begin{cases} \partial_t Y = \Delta Y + a_{11}(x) Y + a_{12}(x) Z + H_1, & \text{in} \quad Q_0, \\ \partial_t Z = \Delta Z + a_{21}(x) Y + a_{22}(x) Z + H_2 & \text{in} \quad Q_0, \\ Y(x,t) = Z(x,t) = 0 & \text{on} \quad \Sigma_0, \\ Y(x,0) = K_1(x), \quad Z(x,0) = K_2(x) & \text{in} \quad \Omega, \end{cases} \quad (3)$$

where H_1 and H_2 are arbitrary functions. Following [4], we can derive a modified Carleman estimate : "shifted Carleman", with a single observation acting on a subdomain ω of Ω for the system (3). Such estimate will be used in the next section to prove stability result for two coefficients with the observation of only one component. Then we can have the following theorem:

Theorem 2.2. *There exist $\lambda_1 = \lambda_1(\Omega, \omega) \geq 1$, $s_1 = s_1(\lambda_1, T) > 1$ and a positive constant $C_1 = C_1(\Omega, \omega, R, T, a_0)$ such that, for any $\lambda \geq \lambda_1$ and any*

$s \geq s_1$ *and* $\epsilon > 0$ *fixed, the following estimate holds:*

$$\lambda^{-4+\epsilon} I(-3, Y) + I(0, Z) \leq C_1 s^4 \lambda^{4+\epsilon} \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 \, dx \, dt \quad (4)$$
$$+ C_1 \left[s^{-3} \lambda^{-4+\epsilon} \iint_Q e^{-2s\eta} \varphi^{-3} \, |H_1|^2 \, dx \, dt + \lambda^{2\epsilon} \iint_Q e^{-2s\eta} \, |H_2|^2 \, dx \, dt \right].$$

Remark 2. *This "shifted Carleman" method allows us to estimate two coefficients, one in each equation, by the observation of only one component with L^2-norm in the observation term. This point is of interest in view to obtain controllability to trajectories with only one control force.*

The proof derives from the proof given in [4, Theorem 2.3]. We only highlight the main points in the proof.

Proof. Using the Carleman estimate (2), the solution $(Y,\, Z)$ of system (3) satisfies

$$\begin{aligned} \lambda^{-4+\epsilon} I(-3,\, Y) + I(0,\, Z) \leq C_0 \Bigg[& \lambda^{\epsilon} \iint_{Q_\omega} e^{-2s\eta} |Y|^2 \, dx \, dt \\ & + s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\eta} \varphi^3 |Z|^2 \, dx \, dt \\ + s^{-3} \lambda^{-4+\epsilon} \iint_Q & e^{-2s\eta} \varphi^{-3} \, (|a_{11} Y|^2 + |a_{12} Z|^2 + |a_{13} H_1|^2) \, dx \, dt \\ & + \iint_Q e^{-2s\eta} \, (|a_{21} Y|^2 + |a_{22} Z|^2 + |a_{23} H_2|^2) \, dx \, dt \Bigg]. \end{aligned} \quad (5)$$

The main difficulty is in estimating the term $I := \lambda^{\epsilon} \iint_{Q_\omega} e^{-2s\eta} |Y|^2 \, dx \, dt$ in function of the localized observation of Z. Thus, through the assumption $a_{21}(x) \geq a_0$ with a_0 is a positive constant, we can estimate the following integral:

$$I' := \lambda^{\epsilon} \iint_{Q_\omega} a_{21} \xi e^{-2s\eta} |Y|^2 \, dx \, dt,$$

where ξ is a smooth cut-function satisfying

$$\begin{cases} \xi(x) = 1 & \forall x \in \omega', \\ 0 < \xi(x) \leq 1 & \forall x \in \omega'', \\ \xi(x) = 0 & \forall x \in \mathbb{R}^n \setminus \omega'', \end{cases}$$

and $\omega' \Subset \omega'' \Subset \omega \Subset \Omega$.

Using the second equation of system (3), we have

$$I' = \lambda^\epsilon \iint_{Q_\omega} e^{-2s\eta}\xi(\partial_t Z - \Delta Z - a_{22}Z - a_{23}H_2)Y \ dx \ dt$$

$$= \lambda^\epsilon \iint_{Q_\omega} e^{-2s\eta}\xi(\partial_t Z)Y \ dx \ dt - \lambda^\epsilon \iint_{Q_\omega} e^{-2s\eta}\xi(\Delta z)Y \ dx \ dt$$

$$\lambda^\epsilon \iint_{Q_\omega} a_{22}e^{-2s\eta}\xi ZY \ dx \ dt - \lambda^\epsilon \iint_{Q_\omega} a_{23}e^{-2s\eta}\xi H_2 Y \ dx \ dt := I_1 + I_2 + I_3 + I_4.$$

Then, by integration by parts and Young inequalities (see [4] for more details), we obtain, for s sufficiently large,

$$|I_1| + |I_2| + |I_3| \leq Cs^4\lambda^{4+\epsilon} \iint_{Q_\omega} e^{-2s\eta}\varphi^4|Z|^2 \ dx \ dt + \text{"absorbed terms"} \ ,$$

and

$$|I_4| \leq C\left(\lambda^{2\epsilon} \iint_{Q_\omega} e^{-2s\eta}|H_2|^2 \ dx \ dt + \iint_{Q_\omega} e^{-2s\eta}|Y|^2 \ dx \ dt\right),$$

C being a generic constant which depends on Ω, ω, R and T and "absorbed terms" means integrals terms dominated by the left hand side of (5) for large s. Finally, we have, for s sufficiently large,

$$\begin{aligned} |I| \leq \frac{1}{a_0}|I'| \ &\leq \ \frac{C}{a_0}s^4\lambda^{4+\epsilon} \iint_{Q_\omega} e^{-2s\eta}\varphi^4|Z|^2 \ dx \ dt + C\lambda^{2\epsilon} \iint_Q e^{-2s\eta}|H_2|^2 \ dx \ dt \\ &+ \ \text{"absorbed terms"} \ . \end{aligned} \tag{6}$$

If we note

$$\begin{aligned} J := s^{-3}\lambda^{-4+\epsilon} &\iint_Q e^{-2s\eta}\varphi^{-3} \ (|a_{11}Y|^2 + |a_{12}Z|^2 + |a_{13}H_1|^2) \ dx \ dt \\ &+ \iint_Q e^{-2s\eta} \ (|a_{21}Y|^2 + |a_{22}Z|^2 + |a_{23}H_2|^2) \ dx \ dt, \end{aligned}$$

we obtain, according (6), for s sufficiently large and $\epsilon > 0$,

$$\begin{aligned} |J| + |I| \ &\leq \ C\left(s^{-3}\lambda^{-4+\epsilon} \iint_Q e^{-2s\eta}\varphi^{-3} \ |H_1|^2 \ dx \ dt + \lambda^{2\epsilon} \iint_Q e^{-2s\eta} \ |H_2|^2 \ dx \ dt\right) \\ &+ \ \text{"absorbed terms"} \ . \end{aligned}$$

Finaly, the previous estimate achieves the proof of Theorem 2.2. $\square$

3 Stability result

In this section we give the proof of Theorem 1.3.
Let (U,V) (resp. $(\widetilde{U},\widetilde{V})$) be solution of (1) associated to (a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, k_1, k_2, U_0, V_0) (resp. (a_{11}, a_{12}, $\widetilde{a}_{13}$, $\widetilde{a}_{21}$, a_{22}, k_1, k_2, U_0, V_0). Then, if we set $u = U-\widetilde{U}$, $v = V-\widetilde{V}$, $Y = \partial_t u$ and $Z = \partial_t v$, (Y,Z) is solution to the following problem

$$\begin{cases} \partial_t Y = \Delta Y + a_{11}(x)Y + a_{12}(x)Z + \gamma_1 \partial_t f(\widetilde{U},\widetilde{V}) + a_{13}(x)\partial_t F(U,V,\widetilde{U},\widetilde{V}), & \text{in} \quad Q_0, \\ \partial_t Z = \Delta Z + a_{21}(x)Y + a_{22}(x)Z + \gamma_2 \partial_t \widetilde{U} & \text{in} \quad Q_0, \\ Y(x,t) = Z(x,t) = 0 & \text{on} \quad \Sigma_0, \\ Y(x,0) = \gamma_1 f(U_0,V_0), \quad Z(x,0) = \gamma_2 U_0 & \text{in} \quad \Omega, \end{cases} \tag{7}$$

where $\gamma_1 = (a_{13}-\widetilde{a}_{13})$, $\gamma_2 = (a_{21}-\widetilde{a}_{21})$ and $F(U,V,\widetilde{U},\widetilde{V}) = f(U,V) - f(\widetilde{U},\widetilde{V})$.
If we apply the modified Carleman estimate (4) to the previous system (7), we have

$$\begin{aligned} \lambda^{-4+\epsilon} I(-3,Y) + I(0,Z) &\le C_1 s^4 \lambda^{4+\epsilon} \iint_{Q_\omega} e^{-2s\eta}\varphi^4 |Z|^2 \, dx \, dt \\ +C_1 \Bigg[s^{-3}\lambda^{-4+\epsilon} &\iint_Q e^{-2s\eta} \, \varphi^{-3} \, (|\gamma_1 \partial_t f(\widetilde{U},\widetilde{V})|^2 + |\partial_t F|^2) \, dx \, dt \\ &+ \lambda^{2\epsilon} \iint_Q e^{-2s\eta} \, |\gamma_2 \partial_t \widetilde{U}|^2 \, dx \, dt \Bigg]. \end{aligned} \tag{8}$$

Now we shall "absorb" the term $A = s^{-3}\lambda^{-4+\epsilon} \iint_Q e^{-2s\eta} \, \varphi^{-3} \, |\partial_t F|^2 \, dx \, dt$. So, we need the following lemma (see [13]):

Lemma 3.1. *There exists a positive constant $C > 0$ such that.*

$$\iint_Q \left| \int_{T'}^t q(x,\xi) d\xi \right|^2 e^{-2s\eta} dx dt \le \frac{C}{s} \iint_Q |q(x,t)|^2 e^{-2s\eta} dx dt$$

for all large $s > 0$ and $q \in L^2(Q)$.

Since $\varphi^{-3} \le C\frac{T^6}{4^3}$, $\varphi^{-3} \le C\frac{T^{12}}{4^6}\varphi^3$ and using Assumption 1.2-*(1)*, the previous Lemma yields

$$A \le C\lambda^{-4+\epsilon} s^{-3}(1+s^{-1}) \iint_Q e^{-2s\eta} \, (|Y|^2 + \varphi^3 |Z|^2) \, dx \, dt. \tag{9}$$

Therefore, for s large enough and $\epsilon > 0$, the integral A can be "absorbed" by the left hand side of (8).

Then (8) can be written as follows

$$
\begin{aligned}
\lambda^{-4+\epsilon} I(-3, Y) + I(0, Z) &\le C_1 s^4 \lambda^{4+\epsilon} \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 \, dx \, dt \\
&+ C_1 \Bigg[s^{-3} \lambda^{-4+\epsilon} \iint_Q e^{-2s\eta} \, \varphi^{-3} \, |\gamma_1 \partial_t f(\widetilde{U}, \widetilde{V})|^2 \, dx \, dt \\
&\qquad + \lambda^{2\epsilon} \iint_Q e^{-2s\eta} \, |\gamma_2 \partial_t \widetilde{U}|^2 \, dx \, dt \Bigg].
\end{aligned} \tag{10}
$$

Let us introduce the following integral

$$
\mathcal{I}_1 = \lambda^{-4+\epsilon} \int_{t_0}^{T'} \int_\Omega M_2^{(-3)} \psi_1 \cdot \psi_1 \, dx \, dt,
$$

with $\psi_1 = e^{-s\eta} Y \varphi^{-3/2}$ and $T' = \frac{T+t_0}{2}$.

We first estimate $\mathcal{I}_1$ with the modified Carleman estimate (10):

$$
\begin{aligned}
\mathcal{I}_1 \quad &\le \quad \frac{1}{2} \lambda^{-2} \left[\lambda^{-4+\epsilon} \| M_2^{(-3)} \psi_1 \|^2_{L^2(Q)} + \lambda^\epsilon \int_{t_0}^{T'} \int_\Omega e^{-2s\eta} \varphi^{-3} |Y|^2 \, dx \, dt \right] \\
&\le \quad \frac{1}{2} \lambda^{-2} C_1 \Bigg[s^4 \lambda^{4+\epsilon} \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 \, dx \, dt \\
&+ \quad s^{-3} \lambda^{-4+\epsilon} \iint_Q e^{-2s\eta} \varphi^{-3} \, |\gamma_1 \partial_t f(\widetilde{U}, \widetilde{V})|^2 \, dx \, dt \\
&+ \quad \lambda^{2\epsilon} \iint_Q e^{-2s\eta} \, |\gamma_2 \partial_t \widetilde{U}|^2 \, dx \, dt \Bigg].
\end{aligned} \tag{11}
$$

By computing $\mathcal{I}_1$, we obtain

$$
\frac{1}{2} \lambda^{-4+\epsilon} \int_\Omega |\psi_1(\cdot\,, T')|^2 \, dx \le 2 |\mathcal{I}_1| + C_1 \lambda^{-3+\epsilon} s(\lambda + 1) \iint_Q e^{-2s\eta} \varphi^{-2} |Y|^2 \, dx \, dt. \tag{12}
$$

Applying the modified Carleman estimate (10) to the last term in (12) with $\varphi^{-2} \le CT^4$ and using the estimate (11), we have

$$
\begin{aligned}
\frac{1}{2} \lambda^{-4+\epsilon} \int_\Omega |\psi_1(\cdot\,, T')|^2 \, dx \quad &\le \quad \frac{1}{2} s \lambda^{-2} C_1 \Bigg[s^4 \lambda^{4+\epsilon} \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 \, dx \, dt \\
&+ \quad s^{-3} \lambda^{-4+\epsilon} \iint_Q e^{-2s\eta} \varphi^{-3} \, |\gamma_1 \partial_t f(\widetilde{U}, \widetilde{V})|^2 \, dx \, dt \\
&+ \quad \lambda^{2\epsilon} \iint_Q e^{-2s\eta} \, |\gamma_2 \partial_t \widetilde{U}|^2 \, dx \, dt \Bigg].
\end{aligned} \tag{13}
$$

Then, the estimate (13) yields

$$\lambda^{-4+\epsilon}\int_\Omega e^{-2s\eta(\cdot\,,\,T')}\varphi^{-3}(\cdot\,,\,T')|Y(\cdot\,,\,T')|^2\;dx \le C\left[s^5\lambda^{2+\epsilon}\iint_{Q_\omega} e^{-2s\eta}\varphi^4|Z|^2\;dx\;dt\right.$$
$$+\; s^{-2}\lambda^{-6+\epsilon}\iint_Q e^{-2s\eta}\;\varphi^{-3}\;|\gamma_1\partial_t f(\widetilde{U},\widetilde{V})|^2\;dx\;dt$$
$$\left.+\; s\lambda^{-2+2\epsilon}\iint_Q e^{-2s\eta}\;|\gamma_2\partial_t\widetilde{U}|^2\;dx\;dt\right].$$

Since $(U,V)(\cdot,T') = (\widetilde{U},\widetilde{V})(\cdot,T')$, we have $Y(\cdot\,,\,T') = \gamma_1 f(\widetilde{U},\widetilde{V})(\cdot\,,\,T')$. Thus, we obtain

$$\begin{aligned}\lambda^{-4+\epsilon}\int_\Omega e^{-2s\eta(\cdot\,,\,T')}\varphi^{-3}(\cdot\,,\,T')|\gamma_1 f(\widetilde{U},\widetilde{V})(\cdot\,,\,T')|^2\;dx&\\ \le C\left[s^5\lambda^{2+\epsilon}\iint_{Q_\omega} e^{-2s\eta}\varphi^4|Z|^2\;dx\;dt\right.&\\ +\; s^{-2}\lambda^{-6+\epsilon}\iint_Q e^{-2s\eta}\;\varphi^{-3}\;|\gamma_1\partial_t f(\widetilde{U},\widetilde{V})|^2\;dx\;dt&\\ \left.+\; s\lambda^{-2+2\epsilon}\iint_Q e^{-2s\eta}\;|\gamma_2\partial_t\widetilde{U}|^2\;dx\;dt\right].&\end{aligned}\tag{14}$$

In a similar way, we introduce

$$\mathcal{I}_2 = \int_{t_0}^{T'}\int_\Omega M_2^{(0)}\psi_2\cdot\psi_2\;dx\;dt,$$

with $\psi_2 = e^{-s\eta}Z$.

Using the fact that $Z(\cdot\,,\,T') = \gamma_2\widetilde{U}(\cdot\,,\,T')$, we obtain

$$\begin{aligned}\int_\Omega e^{-2s\eta(\cdot\,,\,T')}|\gamma_2\widetilde{U}(\cdot\,,\,T')|^2\;dx \le Cs^{5/2}\lambda^{2+\epsilon}\iint_{Q_\omega} e^{-2s\eta}\varphi^4|Z|^2\;dx\;dt&\\ +Cs^{-9/2}\lambda^{-6+\epsilon}\iint_Q e^{-2s\eta}\;\varphi^{-3}\;\;|\gamma_1\partial_t f(\widetilde{U},\widetilde{V})|^2\;dx\;dt&\\ +Cs^{-3/2}\lambda^{-2+2\epsilon}\iint_Q e^{-2s\eta}\;|\gamma_2\partial_t\widetilde{U}|^2\;dx\;dt.&\end{aligned}\tag{15}$$

By adding (14) and (15), we have

$$\begin{aligned}\lambda^{-4+\epsilon}\int_\Omega e^{-2s\eta(\cdot\,,\,T')}\varphi^{-3}(\cdot\,,\,T')|\gamma_1 f(\widetilde{U},\widetilde{V})(\cdot\,,\,T')|^2\;dx+\int_\Omega e^{-2s\eta(\cdot\,,\,T')}|\gamma_2\widetilde{U}(\cdot\,,\,T')|^2\;dx&\\ \le Cs^5\lambda^{2+\epsilon}\iint_{Q_\omega} e^{-2s\eta}\varphi^4|Z|^2\;dx\;dt + Cs\lambda^{-2+2\epsilon}\iint_Q e^{-2s\eta}\;|\gamma_2\partial_t\widetilde{U}|^2\;dx\;dt&\\ +Cs^{-2}\lambda^{-6+\epsilon}\iint_Q e^{-2s\eta}\;\varphi^{-3}\;|\gamma_1\partial_t f(\widetilde{U},\widetilde{V})|^2\;dx\;dt.&\end{aligned}\tag{16}$$

By [14], we can have $\widetilde{U} \in H^1((t_0, T); H^2(\Omega))$ for suitable boundary and initial conditions, so $\partial_t \widetilde{U} \in L^2((t_0, T); H^2(\Omega))$. Moreover, by Assumption 1.2-*(3)* $\partial_t f(\widetilde{U}, \widetilde{V}) \in L^2((t_0, T); H^2(\Omega))$. Then for $n \leq 3$, $\partial_t \widetilde{U}$ and $\partial_t f(\widetilde{U}, \widetilde{V})$ are in $L^2((t_0, T); L^\infty(\Omega))$ by classical Sobolev imbedding. Thus, using Assumptions 1.1-*(2)* and 1.2-*(2)*, the inequality (16), for $0 < \epsilon < 1$ and λ sufficiently large, yields

$$\lambda^{-4+\epsilon} \int_\Omega e^{-2s\eta} \varphi^{-3} |\gamma_1|^2 \, dx + \int_\Omega e^{-2s\eta} |\gamma_2|^2 \, dx \leq C s^5 \lambda^{2+\epsilon} \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 \, dx \, dt.$$

Then, the proof of Theorem 1.3 is complete.

Remark 3. *Denote that in this result we need the use of the two large parameters s and λ. This technical point allows to obtain the observation in L^2-norm.*

We have thus the following uniqueness result:

Corollary 3.2. *Under the same assumptions as in Theorem 1.3 and if*

$$(\partial_t V - \partial_t \widetilde{V})(x, t) = 0 \quad in \quad Q_\omega,$$

then $a_{21} = \widetilde{a}_{21}$ and $a_{13} = \widetilde{a}_{13}$ in Ω.

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